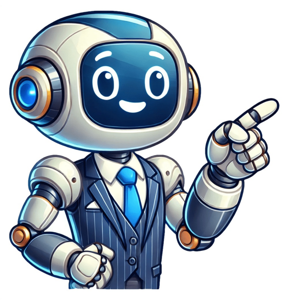

Show Mobile Notice Show All Notes Hide All Notes Mobile Notice You appear to be on a device with a "narrow" screen width (i.e. you are probably on a mobile phone). Due to the nature of the mathematics on this site it is best viewed in landscape mode. If your device is not in landscape mode many of the equations will run off the side of your device (you should be able to scroll/swipe to see them) and some of the menu items will be cut off due to the narrow screen width. It’s now time to start thinking about how to solve nonhomogeneous differential equations. A second order, linear nonhomogeneous differential equation is \[\begin{equation}y'' + p\left(t \right)y' + q\left(t \right)y = g\left(t \right)\label{eq:eq1}\end{equation}\] where \(g(t)\) is a non-zero function. Note that we didn’t go with constant coefficients here because everything that we’re going to do in this section doesn’t require it. Also, we’re using a coefficient of 1 on the second derivative just to make some of the work a little easier to write down. It is not required to be a 1. Before talking about how to solve one of these we need to get some basics out of the way, which is the point of this section. First, we will call \[\begin{equation}y'' + p\left(t \right)y' + q\left(t \right)y = 0\label{eq:eq2}\end{equation}\] the associated homogeneous differential equation to \(\eqref{eq:eq1}\). Now, let’s take a look at the following theorem. Theorem Suppose that \(Y_{1}(t)\) and \(Y_{2}(t)\) are two solutions to \(\eqref{eq:eq1}\) and that \(y_{1}(t)\) and \(y_{2}(t)\) are a fundamental set of solutions to the associated homogeneous differential equation \(\eqref{eq:eq2}\) then, \[{Y_1}\left(t \right) - {Y_2}\left(t \right)\] is a solution to \(\eqref{eq:eq2}\) and it can be written as \[{Y_1}\left(t \right) - {Y_2}\left(t \right) = {c_1}{y_1}\left(t \right) + {c_2}{y_2}\left(t \right)\] Note the notation used here. Capital letters referred to solutions to \(\eqref{eq:eq1}\) while lower case letters referred to solutions to \(\eqref{eq:eq2}\). This is a fairly common convention when dealing with nonhomogeneous differential equations. This theorem is easy enough to prove so let’s do that. To prove that \(Y_{1}(t) - Y_{2}(t)\) is a solution to \(\eqref{eq:eq2}\) all we need to do is plug this into the differential equation and check it. \[\begin{align*}{\left({{Y_1} - {Y_2}} \right)^{\prime \prime }} + p\left(t \right){\left({{Y_1} - {Y_2}} \right)^\prime } + q\left(t \right)\left({{Y_1} - {Y_2}} \right) & = 0\\ {Y_1}^{\prime \prime } + p\left(t \right){Y_1}^\prime + q\left(t \right){Y_1} - \left({{Y_2}^{\prime \prime } + p\left(t \right){Y_2}^\prime + q\left(t \right){Y_2}} \right) & = 0\\ g\left(t \right) - g\left(t \right) & = 0\\ 0 & = 0\end{align*}\] We used the fact that \(Y_{1}(t)\) and \(Y_{2}(t)\) are two solutions to \(\eqref{eq:eq1}\) in the third step. Because they are solutions to \(\eqref{eq:eq1}\) we know that \[\begin{align*}{Y_1}^{\prime \prime } + p\left(t \right){Y_1}^\prime + q\left(t \right){Y_1} & = g\left(t \right)\\ {Y_2}^{\prime \prime } + p\left(t \right){Y_2}^\prime + q\left(t \right){Y_2} & = g\left(t \right)\end{align*}\] So, we were able to prove that the difference of the two solutions is a solution to \(\eqref{eq:eq2}\). Proving that \[{Y_1}\left(t \right) - {Y_2}\left(t \right) = {c_1}{y_1}\left(t \right) + {c_2}{y_2}\left(t \right)\] is even easier. Since \(y_{1}(t)\) and \(y_{2}(t)\) are a fundamental set of solutions to \(\eqref{eq:eq2}\) we know that they form a general solution and so any solution to \(\eqref{eq:eq2}\) can be written in the form \[y\left(t \right) = {c_1}{y_1}\left(t \right) + {c_2}{y_2}\left(t \right)\] Well, \(Y_{1}(t) - Y_{2}(t)\) is a solution to \(\eqref{eq:eq2}\), as we’ve shown above, therefore it can be written as \[{Y_1}\left(t \right) - {Y_2}\left(t \right) = {c_1}{y_1}\left(t \right) + {c_2}{y_2}\left(t \right)\] So, what does this theorem do for us? We can use this theorem to write down the form of the general solution to \(\eqref{eq:eq1}\). Let’s suppose that \(y(t)\) is the general solution to \(\eqref{eq:eq1}\) and that \(Y_{P}(t)\) is any solution to \(\eqref{eq:eq1}\) that we can get our hands on. Then using the second part of our theorem we know that \[y\left(t \right) - {Y_P}\left(t \right) = {c_1}{y_1}\left(t \right) + {c_2}{y_2}\left(t \right)\] where \(y_{1}(t)\) and \(y_{2}(t)\) are a fundamental set of solutions for \(\eqref{eq:eq2}\). Solving for \(y(t)\) gives, \[y\left(t \right) = {c_1}{y_1}\left(t \right) + {c_2}{y_2}\left(t \right) + {Y_P}\left(t \right)\] We will call \[{y_c}\left(t \right) = {c_1}{y_1}\left(t \right) + {c_2}{y_2}\left(t \right)\] the complementary solution and \(Y_{P}(t)\) a particular solution. The general solution to a differential equation can then be written as. \[y\left(t \right) = {y_c}\left(t \right) + {Y_P}\left(t \right)\] So, to solve a nonhomogeneous differential equation, we will need to solve the homogeneous differential equation, \(\eqref{eq:eq2}\), which for constant coefficient differential equations is pretty easy to do, and we’ll need a solution to \(\eqref{eq:eq1}\). This seems to be a circular argument. In order to write down a solution to \(\eqref{eq:eq1}\) we need a solution. However, this isn’t the problem that it seems to be. There are ways to find a solution to \(\eqref{eq:eq1}\). They just won’t, in general, be the general solution. In fact, the next two sections are devoted to exactly that, finding a particular solution to a nonhomogeneous differential equation. There are two common methods for finding particular solutions : Undetermined Coefficients and Variation of Parameters. Both have their advantages and disadvantages as you will see in the next couple of sections. Learning Objectives Write the general solution to a nonhomogeneous differential equation. Solve a nonhomogeneous differential equation by the method of undetermined coefficients. Solve a nonhomogeneous differential equation by the method of variation of parameters. Consider the nonhomogeneous linear differential equation [latex]\large{a_2(x)y^{\prime\prime}+a_1(x)y^\prime+a_0(x)y=r(x)}[/latex]. The associated homogeneous equation [latex]\large{a_2(x)y^{\prime\prime}+a_1(x)y^\prime+a_0(x)y=0}[/latex] is called the complementary equation. We will see that solving the complementary equation is an important step in solving a nonhomogeneous differential equation. A solution [latex]y_p(x)[/latex] of a differential equation that contains no arbitrary constants is called a particular solution to the equation. Let [latex]y_p(x)[/latex] be any particular solution to the nonhomogeneous linear differential equation [latex]\large{a_2(x)y^{\prime\prime}+a_1(x)y^\prime+a_0(x)y=r(x)}[/latex]. Also, let [latex]c_1y_1(x)+c_2y_2(x)[/latex] denote the general solution to the complementary equation. Then, the general solution to the nonhomogeneous equation is given by [latex]\large{y(x)=c_1y_1(x)+c_2y_2(x)+y_p(x)}[/latex]. To prove [latex]y(x)[/latex] is the general solution, we must first show that it solves the differential equation and, second, that any solution to the differential equation can be written in that form. Substituting [latex]y(x)[/latex] into the differential equation, we have [latex]\begin{aligned} a_2(x)y^{\prime\prime}+a_1(x)y^\prime+a_0(x)y&=a_2(x)(c_1y_1+c_2y_2+y_p)^{\prime\prime}+a_1(x)(c_1y_1+c_2y_2+y_p)^\prime+a_0(x)(c_1y_1+c_2y_2+y_p) \\ &=[a_2(x)(c_1y_1+c_2y_2)^{\prime\prime}+a_1(x)(c_1y_1+c_2y_2)^\prime+a_0(x)(c_1y_1+c_2y_2)]+a_2(x)y_p^{\prime\prime}+a_1(x)y_p^\prime+a_0(x)y_p \\ &=0+r(x) \\ &=r(x) \end{aligned}[/latex]. So [latex]y(x)[/latex] is a solution. Now, let [latex]z(x)[/latex] be any solution to [latex]a_2(x)y^{\prime\prime}+a_1(x)y^\prime+a_0(x)y=r(x)[/latex]. Then [latex]\begin{aligned} a_2(x)(z-y_p)^{\prime\prime}+a_1(x)(z-y_p)^\prime+a_0(x)(z-y_p)&=(a_2(x)z^{\prime\prime}+a_1(x)z^\prime+a_0(x)z)-(a_2(x)y_p^{\prime\prime}+a_1(x)y_p^\prime+a_0(x)y_p) \\ &=r(x)-r(x) \\ &=0 \end{aligned}[/latex], so [latex]z(x)-y_p(x)[/latex] is a solution to the complementary equation. But, [latex]c_1y_1(x)+c_2y_2(x)[/latex] is the general solution to the complementary equation, so there are constants [latex]c_1[/latex] and [latex]c_2[/latex] such that [latex]\large{z(x)=y_p(x)=c_1y_1(x)+c_2y_2(x)}[/latex]. Hence, we see that [latex]z(x)=c_1y_1(x)+c_2y_2(x)y_p(x)[/latex]. [latex]_\blacksquare[/latex] Given that [latex]y_p(x)=x[/latex] is a particular solution to the differential equation [latex]y^{\prime\prime}+y=x[/latex], write the general solution and check by verifying that the solution satisfies the equation. Given that [latex]y_p(x)=-2[/latex] is a particular solution to [latex]y^{\prime\prime}-3y^\prime-4y=8[/latex], write the general solution and verify that the general solution satisfies the equation. In the preceding section, we learned how to solve homogeneous equations with constant coefficients. Therefore, for nonhomogeneous equations of the form [latex]ay^{\prime\prime}+by^\prime+cy=r(x)[/latex], we already know how to solve the complementary equation, and the problem boils down to finding a particular solution for the nonhomogeneous equation. We now examine two techniques for this: the method of undetermined coefficients and the method of variation of parameters. The method of undetermined coefficients involves making educated guesses about the form of the particular solution based on the form of [latex]r(x)[/latex]. When we take derivatives of polynomials, exponential functions, sines, and cosines, we get polynomials, exponential functions, sines, and cosines. So when [latex]r(x)[/latex] has one of these forms, it is possible that the solution to the nonhomogeneous differential equation might take that same form. Let’s look at some examples to see how this works. Find the general solution to [latex]y^{\prime\prime}+4y^\prime+3y=3x[/latex]. In Example “Undetermined Coefficients When [latex]r(x)[/latex] Is a Polynomial”, notice that even though [latex]r(x)[/latex] did not include a constant term, it was necessary for us to include the constant term in our guess. If we had assumed a solution of the form [latex]y_p=Ax[/latex] (with no constant term), we would not have been able to find a solution. (Verify this!) If the function [latex]r(x)[/latex] is a polynomial, our guess for the particular solution should be a polynomial of the same degree, and it must include all lower-order terms, regardless of whether they are present in [latex]r(x)[/latex]. Find the general solution to [latex]y^{\prime\prime}-y^\prime-2y=2e^{3x}[/latex]. Find the general solution to [latex]y^{\prime\prime}-4y^\prime+4y=7\sin t-\cos t[/latex]. Watch the following video to see the worked solution to the above Try It You can view the transcript for “CP 7.11” here (opens in new window). In the previous checkpoint, [latex]r(x)[/latex] included both sine and cosine terms. However, even if [latex]r(x)[/latex] included a sine term only or a cosine term only, both terms must be present in the guess. The method of undetermined coefficients also works with products of polynomials, exponentials, sines, and cosines. Some of the key forms of [latex]r(x)[/latex] and the associated guesses for [latex]y_p(x)[/latex] are summarized in Table 7.2 Key Forms for the Method of Undetermined Coefficients below. [latex]r(x)[/latex] Initial guess for [latex]y_p(x)[/latex] [latex]k[/latex] (a constant) [latex]A[/latex] (a constant) [latex]ax+b[/latex] [latex]Ax+B[/latex] (Note: The guess must include both terms even if [latex]b=0[/latex].) [latex]ax^{2}+bx+c[/latex] [latex]Ax^{2}+Bx+C[/latex] (Note: The guess must include all three terms even if [latex]b[/latex] or [latex]c[/latex] are zero.) Higher-order polynomials Polynomial of the same order as [latex]r(x)[/latex] [latex]ae^{\lambda x}[/latex] [latex]Ae^{\lambda x}[/latex] [latex]a\cos\beta x+b\sin\beta x[/latex] [latex]A\cos\beta x+B\sin\beta x[/latex] (Note: The guess must include both terms even if either [latex]a=0[/latex] or [latex]b=0[/latex].) [latex]ae^{\alpha x}\cos\beta x+be^{\alpha x}\sin\beta x[/latex] [latex]Ae^{\alpha x}\cos\beta x+Be^{\alpha x}\sin\beta x[/latex] [latex](ax^2+bx+c)e^{\lambda x}[/latex] [latex](Ax^2+Bx+C)e^{\lambda x}[/latex] [latex](a_2x^2+a_1x+a_0)\cos\beta x+(b_2x^2+b_1x+b_0)\sin\beta x[/latex] [latex](A_2x^2+A_1x+A_0)\cos\beta x+(B_2x^2+B_1x+B_0)\sin\beta x[/latex] [latex](a_2x^2+a_1x+a_0)e^{\alpha x}\cos\beta x+(b_2x^2+b_1x+b_0)e^{\alpha x}\sin\beta x[/latex] [latex](A_2x^2+A_1x+A_0)e^{\alpha x}\cos\beta x+(B_2x^2+B_1x+B_0)e^{\alpha x}\sin\beta x[/latex] Table 7.2 Key Forms for the Method of Undetermined Coefficients Keep in mind that there is a key pitfall to this method. Consider the differential equation [latex]y^{\prime\prime}+5y^\prime+6y=3e^{-2x}[/latex]. Based on the form of [latex]r(x)[/latex], we guess a particular solution of the form [latex]y_p(x)=Ae^{-2x}[/latex]. But when we substitute this expression into the differential equation to find a value for [latex]A[/latex], we run into a problem. We have [latex]\large{y_p^\prime(x)=-2Ae^{-2x}}[/latex] and [latex]\large{y_p^{\prime\prime}=4Ae^{-2x}}[/latex], so we want [latex]\begin{aligned} y^{\prime\prime}+5y^\prime+6y&=3e^{-2x} \\ 4Ae^{-2x}+\left(-2Ae^{-2x}\right)+6Ae^{-2x}&=3e^{-2x} \\ 4Ae^{-2x}-10Ae^{-2x}+6Ae^{-2x}&=3e^{-2x} \\ 0&=3e^{-2x} \end{aligned}[/latex], which is not possible. Looking closely, we see that, in this case, the general solution to the complementary equation is [latex]c_1e^{-2x}+c_2e^{-3x}[/latex]. The exponential function in [latex]r(x)[/latex] is actually a solution to the complementary equation, so, as we just saw, all the terms on the left side of the equation cancel out. We can still use the method of undetermined coefficients in this case, but we have to alter our guess by multiplying it by [latex]x[/latex]. Using the new guess, [latex]y_p(x)=Ae^{-2x}[/latex], we have [latex]\large{y_p^\prime(x)=A\left(e^{-2x}-2xe^{-2x}\right)}[/latex] and [latex]\large{y_p^{\prime\prime}(x)=-4Ae^{-2x}+4Axe^{-2x}}[/latex]. Substitution gives [latex]\begin{aligned} y^{\prime\prime}+5y^\prime+6y&=3e^{-2x} \\ \left(-4Ae^{-2x}+4Axe^{-2x}\right)+5\left(Ae^{-2x}-2Axe^{-2x}\right)+6Axe^{-2x}&=3e^{-2x} \\ -4Ae^{-2x}+4Axe^{-2x}-10Axe^{-2x}+6Axe^{-2x}&=3e^{-2x} \\ Ae^{-2x}&=3e^{-2x} \end{aligned}[/latex], So, [latex]A=3[/latex] and [latex]y_p(x)=3xe^{-2x}[/latex]. This gives us the following general solution [latex]\large{y(x)=c_1e^{-2x}+c_2e^{-2x}+3xe^{-2x}}[/latex]. Note that if [latex]xe^{-2x}[/latex] were also a solution to the complementary equation, we would have to multiply by [latex]x[/latex] again, and we would try [latex]y_p(x)=Ax^2e^{-2x}[/latex]. problem-solving strategy: method of undetermined coefficients Solve the complementary equation and write down the general solution. Based on the form of [latex]r(x)[/latex], make an initial guess for [latex]y_p(x)[/latex]. Check whether any term in the guess for [latex]y_p(x)[/latex] is a solution to the complementary equation. If so, multiply the guess by [latex]x[/latex]. Repeat this step until there are no terms in [latex]y_p(x)[/latex] that solve the complementary equation. Substitute [latex]y_p(x)[/latex] into the differential equation and equate like terms to find values for the unknown coefficients in [latex]y_p(x)[/latex] Add the general solution to the complementary equation and the particular solution you just found to obtain the general solution to the nonhomogeneous equation. Find the general solutions to the following differential equations. [latex]y^{\prime\prime}-9y=-6\cos 3x[/latex] [latex]x^{\prime\prime}+2x'+x=4e^{-t}[/latex] [latex]y^{\prime\prime}-2y^\prime+5y=10^{2}-3x-3[/latex] [latex]y^{\prime\prime}-3y^\prime=-12t[/latex] Find the general solution to the following differential equations. [latex]y^{\prime\prime}-5y^\prime+4y=3e^x[/latex] [latex]y^{\prime\prime}+y^\prime-6y=52\cos2t[/latex] Sometimes, [latex]r(x)[/latex] is not a combination of polynomials, exponentials, or sines and cosines. When this is the case, the method of undetermined coefficients does not work, and we have to use another approach to find a particular solution to the differential equation. We use an approach called the method of variation of parameters. To simplify our calculations a little, we are going to divide the differential equation through by [latex]a[/latex], so we have a leading coefficient of 1. Then the differential equation has the form [latex]y^{\prime\prime}+py^\prime+qy=r(x)[/latex], where [latex]p[/latex] and [latex]q[/latex] are constants. If the general solution to the complementary equation is given by [latex]c_1y_1(x)+c_2y_2(x)[/latex], we are going to look for a particular solution of the form [latex]y_p(x)=u(x)y_1(x)+v(x)y_2(x)[/latex]. In this case, we use the two linearly independent solutions to the complementary equation to form our particular solution. However, we are assuming the coefficients are functions of [latex]x[/latex], rather than constants. We want to find functions [latex]u(x)[/latex] and [latex]v(x)[/latex] such that [latex]y_p(x)[/latex] satisfies the differential equation. We have [latex]\begin{aligned} y_p&=uy_1+vy_2 \\ y_p^\prime&=u^\prime+uy_1^\prime+v^\prime y_2+vy_2^\prime \\ y_p^{\prime\prime}&=(u^\prime y_1+v^\prime y_2)^\prime+u^\prime y_1^\prime+uy_1^{\prime\prime}+v^\prime y_2^\prime+vy_2^{\prime\prime} \end{aligned}[/latex]. Substituting into the differential equation, we obtain [latex]\begin{aligned} y_p^{\prime\prime}+py_p^\prime+qy_p&=[(u^\prime y_1+v^\prime y_2)+u^\prime y_1^\prime+uy_1^{\prime\prime}+v^\prime y_2^\prime+vy_2^{\prime\prime}]+p[u^\prime y_1+uy_1^\prime+v^\prime y_2+vy_2^\prime]+q[uy_1+vy_2] \\ &=u[y_1^{\prime\prime}+py_1^\prime+qy_1]+v[y_2^{\prime\prime}+py_2^\prime+qy_2]+(u^\prime y_1+v^\prime y_2)^\prime+p(u^\prime y_1+v^\prime y_2)+(u^\prime y_1^\prime+v^\prime y_2^\prime) \end{aligned}[/latex]. Note that [latex]y_1[/latex] and [latex]y_2[/latex] are solutions to the complementary equation, so the first two terms are zero. Thus, we have [latex](u^\prime y_1+v^\prime y_2)^\prime+p(u^\prime y_1+v^\prime y_2)+(u^\prime y_1^\prime+v^\prime y_2^\prime)=r(x)[/latex]. If we simplify this equation by imposing the additional condition [latex]u^\prime y_1+v^\prime y_2=0[/latex], the first two terms are zero, and this reduces to [latex]u^\prime y_1^\prime+v^\prime y_2=r(x)[/latex]. So, with this additional condition, we have a system of two equations in two unknowns: [latex]\begin{aligned} u^\prime y_1+v^\prime y_2&=0 \\ u^\prime y_1^\prime+v^\prime y_2^\prime&=r(x) \end{aligned}[/latex]. Solving this system gives us [latex]u'[/latex] and [latex]v'[/latex], which we can integrate to find [latex]u[/latex] and [latex]v[/latex]. Then, [latex]y_p(x)=u(x)y_1(x)+v(x)y_2(x)[/latex] is a particular solution to the differential equation. Solving this system of equations is sometimes challenging, so let’s take this opportunity to review Cramer’s rule, which allows us to solve the system of equations using determinants. The system of equations [latex]\begin{aligned} a_1z_1+b_1z_2&=r_1 \\ a_2z_1+b_2z_2&=r_2 \end{aligned}[/latex] has a unique solution if and only if the determinant of the coefficients is not zero. In this case, the solution is given by [latex]z_1=\frac{\begin{vmatrix}r_1&b_1 \\ r_2&b_2\end{vmatrix}}{\begin{vmatrix}a_1&b_1 \\ a_2&b_2\end{vmatrix}}[/latex] and [latex]z_2=\frac{\begin{vmatrix}a_1&r_1 \\ a_2&r_2\end{vmatrix}}{\begin{vmatrix}a_1&b_1 \\ a_2&b_2\end{vmatrix}}[/latex]. Use Cramer’s rule to solve the following system of equations. [latex]\begin{aligned} x^2z_1+2xz_2&=0 \\ z_1-3x^2z_2&=2x \end{aligned}[/latex]. Use Cramer’s rule to solve the following system of equations. [latex]\begin{aligned} 2xz_1-3z_2&=0 \\ x^2z_1+4xz_2&=x+1 \end{aligned}[/latex]. problem-solving strategy: method of variation of parameters Solve the complementary equation and write down the general solution [latex]c_1y_1(x)+c_2y_2(x)[/latex]. Use Cramer’s rule or another suitable technique to find functions [latex]u^\prime(x)[/latex] and [latex]v^\prime(x)[/latex] satisfying [latex]\begin{aligned} u^\prime y_1+v^\prime y_2&=0 \\ u^\prime y_1^\prime+v^\prime y_2^\prime&=r(x) \end{aligned}[/latex] Integrate [latex]u^\prime[/latex] and [latex]v^\prime[/latex] to find [latex]u(x)[/latex] and [latex]v(x)[/latex]. Then, [latex]y_p(x)=u(x)y_1(x)+v(x)y_2(x)[/latex] is a particular solution to the equation. Add the general solution to the complementary equation and the particular solution found in step 3 to obtain the general solution to the nonhomogeneous equation. Find the general solution to the following differential equations. [latex]y^{\prime\prime}-2y^\prime+y=\frac{e^t}{t^2}[/latex] [latex]y^{\prime\prime}+y=3\sin^2x[/latex] Find the general solution to the following differential equations. [latex]y^{\prime\prime}+y=\sec x[/latex] [latex]x^{\prime\prime}-2x^\prime+x=\frac{e^t}t[/latex] Watch the following videos to see the worked solution to the above Try It You can view the transcript for “CP 7.14a” here (opens in new window). You can view the transcript for “CP 7.14b” here (opens in new window). 7.2.1 Write the general solution to a nonhomogeneous differential equation. 7.2.2 Solve a nonhomogeneous differential equation by the method of undetermined coefficients. 7.2.3 Solve a nonhomogeneous differential equation by the method of variation of parameters. In this section, we examine how to solve nonhomogeneous differential equations. The terminology and methods are different from those we used for homogeneous equations, so let’s start by defining some new terms. Consider the nonhomogeneous linear differential equation a2(x)y″+a1(x)y′+a0(x)y=r(x).a2(x)y″+a1(x)y′+a0(x)y=r(x). The associated homogeneous equation a2(x)y″+a1(x)y′+a0(x)y=0a2(x)y″+a1(x)y′+a0(x)y=0 (7.3) is called the complementary equation. We will see that solving the complementary equation is an important step in solving a nonhomogeneous differential equation. A solution yp(x)yp(x) of a differential equation that contains no arbitrary constants is called a particular solution to the equation. Let yp(x)yp(x) be any particular solution to the nonhomogeneous linear differential equation a2(x)y″+a1(x)y′+a0(x)y=r(x).a2(x)y″+a1(x)y′+a0(x)y=r(x). Also, let c1y1(x)+c2y2(x)c1y1(x)+c2y2(x) denote the general solution to the complementary equation. Then, the general solution to the nonhomogeneous equation is given by y(x)=c1y1(x)+c2y2(x)+yp(x).y(x)=c1y1(x)+c2y2(x)+yp(x). To prove y(x)y(x) is the general solution, we must first show that it solves the differential equation and, second, that any solution to the differential equation can be written in that form. Substituting y(x)y(x) into the differential equation, we have a2(x)y″+a1(x)y′+a0(x)y=a2(x)(c1y1+c2y2+yp)″+a1(x)(c1y1+c2y2+yp)′+a0(x)(c1y1+c2y2+yp)=[a2(x)(c1y1+c2y2)″+a1(x)(c1y1+c2y2)′+a0(x)(c1y1+c2y2)]+a2(x)yp″+a1(x)yp′+a0(x)yp=0+r(x)=r(x).a2(x)y″+a1(x)y′+a0(x)y=a2(x)(c1y1+c2y2+yp)″+a1(x)(c1y1+c2y2+yp)′+a0(x)(c1y1+c2y2+yp)=[a2(x)(c1y1+c2y2)″+a1(x)(c1y1+c2y2)′+a0(x)(c1y1+c2y2)]+a2(x)yp″+a1(x)yp′+a0(x)yp=0+r(x)=r(x). So y(x)y(x) is a solution. Now, let z(x)z(x) be any solution to a2(x)y″+a1(x)y′+a0(x)y=r(x).a2(x)y″+a1(x)y′+a0(x)y=r(x). Then a2(x)(z−yp)″+a1(x)(z−yp)′+a0(x)(z−yp)=(a2(x)z″+a1(x)z′+a0(x)z)−(a2(x)yp″+a1(x)yp′+a0(x)yp)=r(x)−r(x)=0,a2(x)(z−yp)″+a1(x)(z−yp)′+a0(x)(z−yp)=(a2(x)z″+a1(x)z′+a0(x)z)−(a2(x)yp″+a1(x)yp′+a0(x)yp)=r(x)−r(x)=0, so z(x)−yp(x)z(x)−yp(x) is a solution to the complementary equation. But, c1y1(x)+c2y2(x)c1y1(x)+c2y2(x) is the general solution to the complementary equation, so there are constants c1c1 and c2c2 such that z(x)−yp(x)=c1y1(x)+c2y2(x).z(x)−yp(x)=c1y1(x)+c2y2(x). Hence, we see that z(x)=c1y1(x)+c2y2(x)+yp(x).z(x)=c1y1(x)+c2y2(x)+yp(x). □ Given that yp(x)=xyp(x)=x is a particular solution to the differential equation y″+y=x,y″+y=x, write the general solution and check by verifying that the solution satisfies the equation. The complementary equation is y″+y=0,y″+y=0, which has the general solution c1cosx+c2sinx.c1cosx+c2sinx. So, the general solution to the nonhomogeneous equation is y (x) = c 1 cos x + c 2 sin x + x . y (x) = c 1 cos x + c 2 sin x + x . To verify that this is a solution, substitute it into the differential equation. We have y ′ (x) = − c 1 sin x + c 2 cos x + 1 and y ″ (x) = − c 1 cos x − c 2 sin x . y ′ (x) = − c 1 sin x + c 2 cos x + 1 and y ″ (x) = − c 1 cos x − c 2 sin x . Then y ″ (x) + y (x) = − c 1 cos x − c 2 sin x + c 1 cos x + c 2 sin x + x = x . y ″ (x) + y (x) = − c 1 cos x − c 2 sin x + c 1 cos x + c 2 sin x + x = x . So, y(x)y(x) is a solution to y″+y=x.y″+y=x. Given that yp(x)=−2yp(x)=−2 is a particular solution to y″−3y′−4y=8,y″−3y′−4y=8, write the general solution and verify that the general solution satisfies the equation. In the preceding section, we learned how to solve homogeneous equations with constant coefficients. Therefore, for nonhomogeneous equations of the form ay″+by′+cy=r(x),ay″+by′+cy=r(x), we already know how to solve the complementary equation, and the problem boils down to finding a particular solution for the nonhomogeneous equation. We now examine two techniques for this: the method of undetermined coefficients and the method of variation of parameters. The method of undetermined coefficients involves making educated guesses about the form of the particular solution based on the form of r(x).r(x). When we take derivatives of polynomials, exponential functions, sines, and cosines, we get polynomials, exponential functions, sines, and cosines. So when r(x)r(x) has one of these forms, it is possible that the solution to the nonhomogeneous differential equation might take that same form. Let’s look at some examples to see how this works. Find the general solution to y″+4y′+3y=3x.y″+4y′+3y=3x. The complementary equation is y″+4y′+3y=0,y″+4y′+3y=0, with general solution c1e−x+c2e−3x.c1e−x+c2e−3x. Since r(x)=3x,r(x)=3x, the particular solution might have the form yp(x)=Ax+B.yp(x)=Ax+B. If this is the case, then we have yp′(x)=Ayp′(x)=A and yp″(x)=0.yp″(x)=0. For ypyp to be a solution to the differential equation, we must find values for AA and BB such that y ″ + 4 y ′ + 3 y = 3 x 0 + 4 (A) + 3 (A x + B) = 3 x 3 A x + (4 A + 3 B) = 3 x . y ″ + 4 y ′ + 3 y = 3 x 0 + 4 (A) + 3 (A x + B) = 3 x 3 A x + (4 A + 3 B) = 3 x . Setting coefficients of like terms equal, we have 3 A = 3 4 A + 3 B = 0 . 3 A = 3 4 A + 3 B = 0 . Then, A=1A=1 and B=−43,B=−43, so yp(x)=x−43yp(x)=x−43 and the general solution is y (x) = c 1 e − x + c 2 e − 3 x + x − 4 3 . y (x) = c 1 e − x + c 2 e − 3 x + x − 4 3 . In Example 7.12, notice that even though r(x)r(x) did not include a constant term, it was necessary for us to include the constant term in our guess. If we had assumed a solution of the form yp=Axyp=Ax (with no constant term), we would not have been able to find a solution. (Verify this!) If the function r(x)r(x) is a polynomial, our guess for the particular solution should be a polynomial of the same degree, and it must include all lower-order terms, regardless of whether they are present in r(x).r(x). Find the general solution to y″−y′−2y=2e3x.y″−y′−2y=2e3x. The complementary equation is y″−y′−2y=0,y″−y′−2y=0, with the general solution c1e−x+c2e2x.c1e−x+c2e2x. Since r(x)=2e3x,r(x)=2e3x, the particular solution might have the form yp(x)=Ae3x.yp(x)=Ae3x. Then, we have yp′(x)=3Ae3xyp′(x)=3Ae3x and yp″(x)=9Ae3x.yp″(x)=9Ae3x. For ypyp to be a solution to the differential equation, we must find a value for AA such that y ″ − y ′ − 2 y = 2 e 3 x 9 A e 3 x − 3 A e 3 x − 2 A e 3 x = 2 e 3 x 4 A e 3 x = 2 e 3 x . y ″ − y ′ − 2 y = 2 e 3 x 9 A e 3 x − 3 A e 3 x − 2 A e 3 x = 2 e 3 x 4 A e 3 x = 2 e 3 x . So, 4A=24A=2 and A=1/2.A=1/2. Then, yp(x)=(12)e3x,yp(x)=(12)e3x, and the general solution is y (x) = c 1 e − x + c 2 e 2 x + 1 2 e 3 x . y (x) = c 1 e − x + c 2 e 2 x + 1 2 e 3 x . Find the general solution to y″−4y′+4y=7sint−cost.y″−4y′+4y=7sint−cost. In the previous checkpoint, r(x)r(x) included both sine and cosine terms. However, even if r(x)r(x) included a sine term only or a cosine term only, both terms must be present in the guess. The method of undetermined coefficients also works with products of polynomials, exponentials, sines, and cosines. Some of the key forms of r(x)r(x) and the associated guesses for yp(x)yp(x) are summarized in Table 7.2. r(x)r(x) Initial guess for yp(x)yp(x) kk (a constant) AA (a constant) ax+bax+b Ax+BAx+B (Note: The guess must include both terms even if b=0.b=0.) ax2+bx+cax2+bx+c Ax2+Bx+CAx2+Bx+C (Note: The guess must include all three terms even if bb or cc are zero.) Higher-order polynomials Polynomial of the same order as r(x)r(x) aeλxaeλx AeλxAeλx acosβx+bsinβxacosβx+bsinβx Acosβx+BsinβxAcosβx+Bsinβx (Note: The guess must include both terms even if either a=0a=0 or b=0.b=0.) aeαxcosβx+beαxsinβxaeαxcosβx+beαxsinβx Aeαxcosβx+BeαxsinβxAeαxcosβx+Beαxsinβx (ax2+bx+c)eλx(ax2+bx+c)eλx (Ax2+Bx+C)eλx(Ax2+Bx+C)eλx (a2x2+a1x+a0)cosβx+(b2x2+b1x+b0)sinβx(a2x2+a1x+a0)cosβx+(b2x2+b1x+b0)sinβx (A2x2+A1x+A0)cosβx+(B2x2+B1x+B0)sinβx(A2x2+A1x+A0)cosβx+(B2x2+B1x+B0)sinβx (a2x2+a1x+a0)eαxcosβx+(b2x2+b1x+b0)eαxsinβx(a2x2+a1x+a0)eαxcosβx+(b2x2+b1x+b0)eαxsinβx (A2x2+A1x+A0)eαxcosβx+(B2x2+B1x+B0)eαxsinβx(A2x2+A1x+A0)eαxcosβx+(B2x2+B1x+B0)eαxsinβx Table 7.2 Key Forms for the Method of Undetermined Coefficients Keep in mind that there is a key pitfall to this method. Consider the differential equation y″+5y′+6y=3e−2x.y″+5y′+6y=3e−2x. Based on the form of r(x),r(x), we guess a particular solution of the form yp(x)=Ae−2x.yp(x)=Ae−2x. But when we substitute this expression into the differential equation to find a value for A,A, we run into a problem. We have yp′(x)=−2Ae−2xyp′(x)=−2Ae−2x and yp″=4Ae−2x,yp″=4Ae−2x, so we want y″+5y′+6y=3e−2x4Ae−2x+5(−2Ae−2x)+6Ae−2x=3e−2x4Ae−2x−10Ae−2x+6Ae−2x=3e−2x0=3e−2x,y″+5y′+6y=3e−2x4Ae−2x+5(−2Ae−2x)+6Ae−2x=3e−2x4Ae−2x−10Ae−2x+6Ae−2x=3e−2x0=3e−2x, which is not possible. Looking closely, we see that, in this case, the general solution to the complementary equation is c1e−2x+c2e−3x.c1e−2x+c2e−3x. The exponential function in r(x)r(x) is actually a solution to the complementary equation, so, as we just saw, all the terms on the left side of the equation cancel out. We can still use the method of undetermined coefficients in this case, but we have to alter our guess by multiplying it byx.x. Using the new guess, yp(x)=Axe−2x,yp(x)=Axe−2x, we have yp′(x)=A(e−2x−2xe−2x)yp′(x)=A(e−2x−2xe−2x) and yp″(x)=−4Ae−2x+4Axe−2x.yp″(x)=−4Ae−2x+4Axe−2x. Substitution gives y″+5y′+6y=3e−2x(−4Ae−2x+4Axe−2x)+5(Ae−2x−2Axe−2x)+6Axe−2x=3e−2x−4Ae−2x+4Axe−2x+5Ae−2x−10Axe−2x+6Axe−2x=3e−2xAe−2x=3e−2x.y″+5y′+6y=3e−2x(−4Ae−2x+4Axe−2x)+5(Ae−2x−2Axe−2x)+6Axe−2x=3e−2x−4Ae−2x+4Axe−2x+5Ae−2x−10Axe−2x+6Axe−2x=3e−2xAe−2x=3e−2x. So, A=3A=3 and yp(x)=3xe−2x.yp(x)=3xe−2x. This gives us the following general solution y(x)=c1e−2x+c2e−3x+3xe−2x.y(x)=c1e−2x+c2e−3x+3xe−2x. Note that if xe−2xxe−2x were also a solution to the complementary equation, we would have to multiply by xx again, and we would try yp(x)=Ax2e−2x.yp(x)=Ax2e−2x. Solve the complementary equation and write down the general solution. Based on the form of r(x),r(x), make an initial guess for yp(x).yp(x). Check whether any term in the guess for yp(x)yp(x) is a solution to the complementary equation. If so, multiply the guess byx.x. Repeat this step until there are no terms in yp(x)yp(x) that solve the complementary equation. Substitute yp(x)yp(x) into the differential equation and equate like terms to find values for the unknown coefficients in yp(x).yp(x). Add the general solution to the complementary equation and the particular solution you just found to obtain the general solution to the nonhomogeneous equation. Find the general solutions to the following differential equations. y″−9y=−6cos3xy″−9y=−6cos3x x″+2x′+x=4e−tx″+2x′+x=4e−t y″−2y′+5y=10x2−3x−3y″−2y′+5y=10x2−3x−3 y″−3y′=−12ty″−3y′=−12t The complementary equation is y″−9y=0,y″−9y=0, which has the general solution c1e3x+c2e−3xc1e3x+c2e−3x (step 1). Based on the form of r(x)=−6cos3x,r(x)=−6cos3x, our initial guess for the particular solution is yp(x)=Acos3x+Bsin3xyp(x)=Acos3x+Bsin3x (step 2). None of the terms in yp(x)yp(x) solve the complementary equation, so this is a valid guess (step 3). Now we want to find values for AA and B,B, so substitute ypyp into the differential equation. We have yp′(x)=−3Asin3x+3Bcos3xandyp″(x)=−9Acos3x−9Bsin3x,yp′(x)=−3Asin3x+3Bcos3xandyp″(x)=−9Acos3x−9Bsin3x, so we want to find values of AA and BB such that y″−9y=−6cos3x−9Acos3x−9Bsin3x−9(Acos3x+Bsin3x)=−6cos3x−18Acos3x−18Bsin3x=−6cos3x.y″−9y=−6cos3x−9Acos3x−9Bsin3x−9(Acos3x+Bsin3x)=−6cos3x−18Acos3x−18Bsin3x=−6cos3x. Therefore, −18A=−6−18B=0.−18A=−6−18B=0. This gives A=13A=13 and B=0,B=0, so yp(x)=(13)cos3xyp(x)=(13)cos3x (step 4). Putting everything together, we have the general solution y(x)=c1e3x+c2e−3x+13cos3x.y(x)=c1e3x+c2e−3x+13cos3x. The complementary equation is x″+2x′+x=0,x″+2x′+x=0, which has the general solution c1e−t+c2te−tc1e−t+c2te−t (step 1). Based on the form r(t)=4e−t,r(t)=4e−t, our initial guess for the particular solution is xp(t)=Ae−txp(t)=Ae−t (step 2). However, we see that this guess solves the complementary equation, so we must multiply byt,t, which gives a new guess: xp(t)=Ate−txp(t)=Ate−t (step 3). Checking this new guess, we see that it, too, solves the complementary equation, so we must multiply by t again, which gives xp(t)=At2e−txp(t)=At2e−t (step 3 again). Now, checking this guess, we see that xp(t)xp(t) does not solve the complementary equation, so this is a valid guess (step 3 yet again). We now want to find a value for A,A, so we substitute xpxp into the differential equation. We have xp(t)=At2e−t,soxp′(t)=2Ate−t−At2e−txp(t)=At2e−t,soxp′(t)=2Ate−t−At2e−t and xp″(t)=2Ae−t−2Ate−t−(2Ate−t−At2e−t)=2Ae−t−4Ate−t+At2e−t.xp″(t)=2Ae−t−2Ate−t−(2Ate−t−At2e−t)=2Ae−t−4Ate−t+At2e−t. Substituting into the differential equation, we want to find a value of AA so that x″+2x′+x=4e−t2Ae−t−4Ate−t+At2e−t+2(2Ate−t−At2e−t)+At2e−t=4e−t2Ae−t=4e−t.x″+2x′+x=4e−t2Ae−t−4Ate−t+At2e−t+2(2Ate−t−At2e−t)+At2e−t=4e−t2Ae−t=4e−t. This gives A=2,A=2, so xp(t)=2t2e−txp(t)=2t2e−t (step 4). Putting everything together, we have the general solution x(t)=c1e−t+c2te−t+2t2e−t.x(t)=c1e−t+c2te−t+2t2e−t. The complementary equation is y″−2y′+5y=0,y″−2y′+5y=0, which has the general solution c1excos2x+c2exsin2xc1excos2x+c2exsin2x (step 1). Based on the form r(x)=10x2−3x−3,r(x)=10x2−3x−3, our initial guess for the particular solution is yp(x)=Ax2+Bx+Cyp(x)=Ax2+Bx+C (step 2). None of the terms in yp(x)yp(x) solve the complementary equation, so this is a valid guess (step 3). We now want to find values for A,A, B,B, and C,C, so we substitute ypyp into the differential equation. We have yp′(x)=2Ax+Byp′(x)=2Ax+B and yp″(x)=2A,yp″(x)=2A, so we want to find values of A, B, and CC such that y″−2y′+5y=10x2−3x−32A−2(2Ax+B)+5(Ax2+Bx+C)=10x2−3x−35Ax2+(5B−4A)x+(5C−2B+2A)=10x2−3x−3.y″−2y′+5y=10x2−3x−32A−2(2Ax+B)+5(Ax2+Bx+C)=10x2−3x−35Ax2+(5B−4A)x+(5C−2B+2A)=10x2−3x−3. Therefore, 5A=105B−4A=−35C−2B+2A=−3.5A=105B−4A=−35C−2B+2A=−3. This gives A=2,A=2, B=1,B=1, and C=−1,C=−1, so yp(x)=2x2+x−1yp(x)=2x2+x−1 (step 4). Putting everything together, we have the general solution y(x)=c1excos2x+c2exsin2x+2x2+x−1.y(x)=c1excos2x+c2exsin2x+2x2+x−1. The complementary equation is y″−3y′=0,y″−3y′=0, which has the general solution c1e3t+c2c1e3t+c2 (step 1). Based on the form r(t)=−12t,r(t)=−12t, our initial guess for the particular solution is yp(t)=At+Byp(t)=At+B (step 2). However, we see that the constant term in this guess solves the complementary equation, so we must multiply by t,t, which gives a new guess: yp(t)=At2+Btyp(t)=At2+Bt (step 3). Checking this new guess, we see that none of the terms in yp(t)yp(t) solve the complementary equation, so this is a valid guess (step 3 again). We now want to find values for AA and B,B, so we substitute ypyp into the differential equation. We have yp′(t)=2At+Byp′(t)=2At+B and yp″(t)=2A,yp″(t)=2A, so we want to find values of AA and BB such that y″−3y′=−12t2A−3(2At+B)=−12t−6At+(2A−3B)=−12t.y″−3y′=−12t2A−3(2At+B)=−12t−6At+(2A−3B)=−12t. Therefore, −6A=−122A−3B=0.−6A=−122A−3B=0. This gives A=2A=2 and B=4/3,B=4/3, so yp(t)=2t2+(4/3)typ(t)=2t2+(4/3)t (step 4). Putting everything together, we have the general solution y(t)=c1e3t+c2+2t2+43t.y(t)=c1e3t+c2+2t2+43t. Find the general solution to the following differential equations. y″−5y′+4y=3exy″−5y′+4y=3ex y″+y′−6y=52cos2ty″+y′−6y=52cos2t Sometimes, r(x)r(x) is not a combination of polynomials, exponentials, or sines and cosines. When this is the case, the method of undetermined coefficients does not work, and we have to use another approach to find a particular solution to the differential equation. We use an approach called the method of variation of parameters. To simplify our calculations a little, we are going to divide the differential equation through by a,a, so we have a leading coefficient of 1. Then the differential equation has the form y″+py′+qy=r(x),y″+py′+qy=r(x), where pp and qq are constants. If the general solution to the complementary equation is given by c1y1(x)+c2y2(x),c1y1(x)+c2y2(x), we are going to look for a particular solution of the form yp(x)=u(x)y1(x)+v(x)y2(x).yp(x)=u(x)y1(x)+v(x)y2(x). In this case, we use the two linearly independent solutions to the complementary equation to form our particular solution. However, we are assuming the coefficients are functions of x, rather than constants. We want to find functions u(x)u(x) and v(x)v(x) such that yp(x)yp(x) satisfies the differential equation. We have yp=uy1+vy2yp′=u′y1+uy1′+v′y2+vy2′yp″=(u′y1+v′y2)′+u′y1′+uy1″+v′y2′+vy2″.yp=uy1+vy2yp′=u′y1+uy1′+v′y2+vy2′yp″=(u′y1+v′y2)′+u′y1′+uy1″+v′y2′+vy2″. Substituting into the differential equation, we obtain yp″+pyp′+qyp=[(u′y1+v′y2)′+u′y1′+uy1″+v′y2′+vy2″]+p[u′y1+uy1′+v′y2+vy2′]+q[uy1+vy2]=u[y1″+py1′+qy1]+v[y2″+py2′+qy2]+(u′y1+v′y2)′+p(u′y1+v′y2)+(u′y1′+v′y2′).yp″+pyp′+qyp=[(u′y1+v′y2)′+u′y1′+uy1″+v′y2′+vy2″]+p[u′y1+uy1′+v′y2+vy2′]+q[uy1+vy2]=u[y1″+py1′+qy1]+v[y2″+py2′+qy2]+(u′y1+v′y2)′+p(u′y1+v′y2)+(u′y1′+v′y2′). Note that y1y1 and y2y2 are solutions to the complementary equation, so the first two terms are zero. Thus, we have (u′y1+v′y2)′+p(u′y1+v′y2)+(u′y1′+v′y2′)=r(x).(u′y1+v′y2)′+p(u′y1+v′y2)+(u′y1′+v′y2′)=r(x). If we simplify this equation by imposing the additional condition u′y1+v′y2=0,u′y1+v′y2=0, the first two terms are zero, and this reduces to u′y1′+v′y2′=r(x).u′y1′+v′y2′=r(x). So, with this additional condition, we have a system of two equations in two unknowns: u′y1+v′y2=0u′y1′+v′y2′=r(x).u′y1+v′y2=0u′y1′+v′y2′=r(x). Solving this system gives us u′u′ and v′,v′, which we can integrate to find u and v. Then, yp(x)=u(x)y1(x)+v(x)y2(x)yp(x)=u(x)y1(x)+v(x)y2(x) is a particular solution to the differential equation. Solving this system of equations is sometimes challenging, so let’s take this opportunity to review Cramer’s rule, which allows us to solve the system of equations using determinants. The system of equations a1z1+b1z2=r1a2z1+b2z2=r2a1z1+b1z2=r1a2z1+b2z2=r2 has a unique solution if and only if the determinant of the coefficients is not zero. In this case, the solution is given by z1=|r1b1r2b2||a1b1a2b2|andz2=|a1r1a2r2||a1b1a2b2|.z1=|r1b1r2b2||a1b1a2b2|andz2=|a1r1a2r2||a1b1a2b2|. Use Cramer’s rule to solve the following system of equations. x2z1+2xz2=0z1−3x2z2=2xx2z1+2xz2=0z1−3x2z2=2x We have a 1 (x) = x 2 a 2 (x) = 1 b 1 (x) = 2 x b 2 (x) = −3 x 2 r 1 (x) = 0 r 2 (x) = 2 x . a 1 (x) = x 2 a 2 (x) = 1 b 1 (x) = 2 x b 2 (x) = −3 x 2 r 1 (x) = 0 r 2 (x) = 2 x . Then, | a 1 b 1 a 2 b 2 | = | x 2 2 x 1 −3 x 2 | = −3 x 4 − 2 x | a 1 b 1 a 2 b 2 | = | x 2 2 x 1 −3 x 2 | = −3 x 4 − 2 x and | r 1 b 1 r 2 b 2 | = | 0 2 x 2 x −3 x 2 | = 0 − 4 x 2 = −4 x 2 . | r 1 b 1 r 2 b 2 | = | 0 2 x 2 x −3 x 2 | = 0 − 4 x 2 = −4 x 2 . Thus, z 1 = | r 1 b 1 r 2 b 2 | | a 1 b 1 a 2 b 2 | = −4 x 2 −3 x 4 − 2 x = 4 x 3 x 3 + 2 . z 1 = | r 1 b 1 r 2 b 2 | | a 1 b 1 a 2 b 2 | = −4 x 2 −3 x 4 − 2 x = 4 x 3 x 3 + 2 . In addition, | a 1 r 1 a 2 r 2 | = | x 2 0 1 2 x | = 2 x 3 − 0 = 2 x 3 . | a 1 r 1 a 2 r 2 | = | x 2 0 1 2 x | = 2 x 3 − 0 = 2 x 3 . Thus, z 2 = | a 1 r 1 a 2 r 2 | | a 1 b 1 a 2 b 2 | = 2 x 3 −3 x 4 − 2 x = −2 x 2 3 x 3 + 2 . z 2 = | a 1 r 1 a 2 r 2 | | a 1 b 1 a 2 b 2 | = 2 x 3 −3 x 4 − 2 x = −2 x 2 3 x 3 + 2 . Use Cramer’s rule to solve the following system of equations. 2xz1−3z2=0x2z1+4xz2=x+12xz1−3z2=0x2z1+4xz2=x+1 Solve the complementary equation and write down the general solution c1y1(x)+c2y2(x).c1y1(x)+c2y2(x). Use Cramer’s rule or another suitable technique to find functions u′(x)u′(x) and v′(x)v′(x) satisfying u′y1+v′y2=0u′y1′+v′y2′=r(x).u′y1+v′y2=0u′y1′+v′y2′=r(x). Integrate u′u′ and v′v′ to find u(x)u(x) and v(x).v(x). Then, yp(x)=u(x)y1(x)+v(x)y2(x)yp(x)=u(x)y1(x)+v(x)y2(x) is a particular solution to the equation. Add the general solution to the complementary equation and the particular solution found in step 3 to obtain the general solution to the nonhomogeneous equation. Find the general solution to the following differential equations. y″−2y′+y=ett2y″−2y′+y=ett2 y″+y=3sin2xy″+y=3sin2x The complementary equation is y″−2y′+y=0y″−2y′+y=0 with associated general solution c1et+c2tet.c1et+c2tet. Therefore, y1(t)=ety1(t)=et and y2(t)=tet.y2(t)=tet. Calculating the derivatives, we get y1′(t)=ety1′(t)=et and y2′(t)=et+tety2′(t)=et+tet (step 1). Then, we want to find functions u′(t)u′(t) and v′(t)v′(t) so that u′et+v′tet=0u′et+v′(et+tet)=ett2.u′et+v′tet=0u′et+v′(et+tet)=ett2. Applying Cramer’s rule, we have u′=|0tetett2et+tet||ettetetet+tet|=0−tet(ett2)et(et+tet)−ettet=−e2tte2t=−1tu′=|0tetett2et+tet||ettetetet+tet|=0−tet(ett2)et(et+tet)−ettet=−e2tte2t=−1t and v′=|et0etett2||ettetetet+tet|=et(ett2)e2t=1t2(step 2).v′=|et0etett2||ettetetet+tet|=et(ett2)e2t=1t2(step 2). Integrating, we get u=−∫1tdt=−ln|t|v=∫1t2dt=−1t(step 3).u=−∫1tdt=−ln|t|v=∫1t2dt=−1t(step 3). Then we have yp=−etln|t|−1ttet=−etln|t|−et(step 4).yp=−etln|t|−1ttet=−etln|t|−et(step 4). The etet term is a solution to the complementary equation, so we don’t need to carry that term into our general solution explicitly. The general solution is y(t)=c1et+c2tet−etln|t|(step 5).y(t)=c1et+c2tet−etln|t|(step 5). The complementary equation is y″+y=0y″+y=0 with associated general solution c1cosx+c2sinx.c1cosx+c2sinx. So, y1(x)=cosxy1(x)=cosx and y2(x)=sinxy2(x)=sinx (step 1). Then, we want to find functions u′(x)u′(x) and v′(x)v′(x) such that u′cosx+v′sinx=0−u′sinx+v′cosx=3sin2x.u′cosx+v′sinx=0−u′sinx+v′cosx=3sin2x. Applying Cramer’s rule, we have u′=|0sinx3sin2xcosx||cosxsinx−sinxcosx|=0−3sin3xcos2x+sin2x=−3sin3xu′=|0sinx3sin2xcosx||cosxsinx−sinxcosx|=0−3sin3xcos2x+sin2x=−3sin3x and v′=|cosx0−sinx3sin2x||cosxsinx−sinxcosx|=3sin2xcosx1=3sin2xcosx(step 2).v′=|cosx0−sinx3sin2x||cosxsinx−sinxcosx|=3sin2xcosx1=3sin2xcosx(step 2). Integrating first to find u, we get u=∫−3sin3xdx=−3[−13sin2xcosx+23∫sinxdx]=sin2xcosx+2cosx.u=∫−3sin3xdx=−3[−13sin2xcosx+23∫sinxdx]=sin2xcosx+2cosx. Now, we integrate to find v. Using substitution (with w=sinxw=sinx), we get v=∫3sin2xcosxdx=∫3w2dw=w3=sin3x.v=∫3sin2xcosxdx=∫3w2dw=w3=sin3x. Then, yp =(sin2xcosx+2cosx)cosx+(sin3x)sinx=sin2xcos2x+2cos2x+sin4x =2cos2x+sin2x(cos2x+sin2x)(step 3).=2cos2x+sin2x=cos2x+1yp =(sin2xcosx+2cosx)cosx+(sin3x)sinx=sin2xcos2x+2cos2x+sin4x =2cos2x+sin2x(cos2x+sin2x)(step 3).=2cos2x+sin2x=cos2x+1 The general solution is y(x)=c1cosx+c2sinx+1+cos2x(step 4).y(x)=c1cosx+c2sinx+1+cos2x(step 4). Find the general solution to the following differential equations. y″+y=secxy″+y=secx x″−2x′+x=ettx″−2x′+x=ett Solve the following equations using the method of undetermined coefficients. 54. 2 y ″ − 5 y ′ − 12 y = 6 2 y ″ − 5 y ′ − 12 y = 6 55. 3 y ″ + y ′ − 4 y = 8 3 y ″ + y ′ − 4 y = 8 56. y ″ − 6 y ′ + 5 y = e − x y ″ − 6 y ′ + 5 y = e − x 57. y ″ + 16 y = e −2 x y ″ + 16 y = e −2 x 58. y ″ − 4 y = x 2 + 1 y ″ − 4 y = x 2 + 1 59. y ″ − 4 y ′ + 4 y = 8 x 2 + 4 x y ″ − 4 y ′ + 4 y = 8 x 2 + 4 x 60. y ″ − 2 y ′ − 3 y = sin 2 x y ″ − 2 y ′ − 3 y = sin 2 x 61. y ″ + 2 y ′ + y = sin x + cos x y ″ + 2 y ′ + y = sin x + cos x 62. y ″ + 9 y = e x cos x y ″ + 9 y = e x cos x 63. y ″ + y = 3 sin 2 x + x cos 2 x y ″ + y = 3 sin 2 x + x cos 2 x 64. y ″ + 3 y ′ − 28 y = 10 e 4 x y ″ + 3 y ′ − 28 y = 10 e 4 x 65. y ″ + 10 y ′ + 25 y = x e −5 x + 4 y ″ + 10 y ′ + 25 y = x e −5 x + 4 In each of the following problems, Write the form for the particular solution yp(x)yp(x) for the method of undetermined coefficients. [T] Use a computer algebra system to find a particular solution to the given equation. 66. y ″ − y ′ − y = x + e − x y ″ − y ′ − y = x + e − x 67. y ″ − 3 y = x 2 − 4 x + 11 y ″ − 3 y = x 2 − 4 x + 11 68. y ″ − y ′ − 4 y = e x cos 3 x y ″ − y ′ − 4 y = e x cos 3 x 69. 2 y ″ − y ′ + y = (x 2 − 5 x) e − x 2 y ″ − y ′ + y = (x 2 − 5 x) e − x 70. 4 y ″ + 5 y ′ − 2 y = e 2 x + x sin x 4 y ″ + 5 y ′ − 2 y = e 2 x + x sin x 71. y ″ − y ′ − 2 y = x 2 e x sin x y ″ − y ′ − 2 y = x 2 e x sin x Solve the differential equation using either the method of undetermined coefficients or the variation of parameters. 72. y ″ + 3 y ′ − 4 y = 2 e x y ″ + 3 y ′ − 4 y = 2 e x 73. y ″ + 2 y ′ = e 3 x y ″ + 2 y ′ = e 3 x 74. y ″ + 6 y ′ + 9 y = e − x y ″ + 6 y ′ + 9 y = e − x 75. y ″ + 2 y ′ − 8 y = 6 e 2 x y ″ + 2 y ′ − 8 y = 6 e 2 x Solve the differential equation using the method of variation of parameters. 76. 4 y ″ + y = 2 sin x 4 y ″ + y = 2 sin x 77. y ″ − 9 y = 8 x y ″ − 9 y = 8 x 78. y ″ + y = sec x , 0 < x < π / 2 y ″ + y = sec x , 0 < x < π / 2 79. y ″ + 4 y = 3 csc 2 x , 0 < x < π / 2 y ″ + 4 y = 3 csc 2 x , 0 < x < π / 2 Find the unique solution satisfying the differential equation and the initial conditions given, where yp(x)yp(x) is the particular solution. 80. y″−2y′+y=12ex,y″−2y′+y=12ex, yp(x)=6x2ex,yp(x)=6x2ex, y(0)=6,y′(0)=0y(0)=6,y′(0)=0 81. y″−7y′=4xe7x,y″−7y′=4xe7x, yp(x)=27x2e7x−449xe7x,yp(x)=27x2e7x−449xe7x, y(0)=−1,y′(0)=0y(0)=−1,y′(0)=0 82. y″+y=cosx−4sinx,y″+y=cosx−4sinx, yp(x)=2xcosx+12xsinx,yp(x)=2xcosx+12xsinx, y(0)=8,y′(0)=−4y(0)=8,y′(0)=−4 83. y″−5y′=e5x+8e−5x,y″−5y′=e5x+8e−5x, yp(x)=15xe5x+425e−5x,yp(x)=15xe5x+425e−5x, y(0)=−2,y′(0)=0y(0)=−2,y′(0)=0 In each of the following problems, two linearly independent solutions—y1y1 and y2y2—are given that satisfy the corresponding homogeneous equation. Use the method of variation of parameters to find a particular solution to the given nonhomogeneous equation. Assume x > 0 in each exercise. 84. x2y″+2xy′−2y=3x,x2y″+2xy′−2y=3x, y1(x)=x,y2(x)=x−2y1(x)=x,y2(x)=x−2 85. x2y″−2y=10x2−1,x2y″−2y=10x2−1, y1(x)=x2,y2(x)=x−1y1(x)=x2,y2(x)=x−1 Learning about non-homogeneous differential equations is fundamental since there are instances when we’re given complex equations with functions on both sides of the equation. Laws of motion, for example, rely on non-homogeneous differential equations, so it is important that we learn how to solve these types of equations.Non-homogeneous differential equations contain functions on the right-hand side of the equations. We can find their solutions by writing down the general solution of the associated homogeneous differential equation and the particular solution of the non-homogeneous term.Solving non-homogeneous differential equations will still require our knowledge on solving second order homogeneous differential equations, so keep your notes handy on characteristic and second order homogeneous equations. This article covers the fundamentals needed to identify non-homogeneous differential equations and two important methods that will help you find their solutions.What Is a Non Homogeneous Differential Equation?Non-homogeneous differential equations are simply differential equations that do not satisfy the conditions for homogeneous equations. In the past, we’ve learned that homogeneous equations are equations that have zero on the right-hand side of the equation.This means that non-homogeneous differential equations are differential equations that have a function on the right-hand side of their equation.Here are some examples of homogeneous and non-homogenous differential equations. Through these examples, we’ll learn how to identify differential equations based on their form.Homogeneous Differential EquationsNon-Homogenous Differential Equations\begin{aligned}y^{\prime \prime} + y^{\prime} – 2y = 0\end{aligned}\begin{aligned}y^{\prime \prime} + y^{\prime} – 2y = 4x\end{aligned}\begin{aligned}y^{\prime \prime} -3y^{\prime} + y = 0\end{aligned}\begin{aligned}y^{\prime \prime} -3y^{\prime} + y = -6 - 2x + 3x^2\end{aligned}\begin{aligned}y^{\prime \prime \prime} +4y^{\prime\prime} + 4y^{\prime} + y = 0\end{aligned}\begin{aligned}y^{\prime \prime \prime} +4y^{\prime\prime} + 4y^{\prime} + y = 2e^x\end{aligned}From these three equations alone, we can clearly see that the right-hand side of the equations will determine whether the differential equations are homogeneous or non-homogeneous. We call the equations on the left column the associated homogeneous equations, since they share identical expressions with their non-homogeneous counterparts. Since we’ve been working on first order and second order non-homogeneous linear differential equations, let us show you their general forms:\begin{aligned}\textbf{First Order} &: y^{\prime} + P(x)y = f(x)\\\textbf{Second Order} &: y^{\prime} + P(x)y^{\prime} + Q(x)y = g(x)\\ \end{aligned}In the next section, we’ll show you how to solve these types of equations by applying old techniques and even new methods!How To Solve Non Homogeneous Differential Equations?We can solve non-homogeneous linear differential equations by finding the general solution of the associated homogeneous differential equation, y_h, and the particular solution of the non-homogeneous equation, y_p.Let’s work on a first order non-homogeneous differential equation to start this section lightly.\begin{aligned}y^{\prime} + P(x)y = f(x)\end{aligned}In the past, we’ve learned how to deal with these types of equations. We’ve shown you how to use integrating factors to write the general equation for a first order non-homogeneous differential equation. This process will eventually lead to a general solution shown below.\begin{aligned}y &= h(x) + p(x) \end{aligned}For this solution, $g(x)$ represents the general solution, while $h(x)$ represents the particular solution of the first order non-homogeneous differential equation. Since we’ve set a special article for that, our discussion will focus on second order non-homogeneous differential equations and those with a higher order as well. SOLUTION FOR NON HOMOGENEOUS DIFFERENTIAL EQUATIONSSuppose that we have a second order non-homogeneous linear differential equation shown below.\begin{aligned}y^{\prime \prime} +ay^{\prime} + by &= g(x)\end{aligned}For this equation, a and b are constants. When this happens, the nonhomogeneous differential equation with a general solution as shown below.\begin{aligned}\boldsymbol{y} &= \boldsymbol{y_h + y_p}\\\\y_h&: \text{General Solution}\\y_p&: \text{Particular Solution}\end{aligned}This can be extended with nth-order non-homogeneous linear differential equation. The general solution of this linear differential equation is as shown below.\begin{aligned}c_1y_1 + c_2\end{aligned}In previous articles, we have also learned how to find the general solutions for homogeneous differential equations. We’ll apply the same methods and techniques when finding the general solution for our associated homogeneous equation. This means that we’ll be focusing on techniques to find the particular solution for these non-homogeneous equations.How To Find the Particular Solution of a Non Homogeneous Differential EquationThe two most common methods when finding the particular solution of a non-homogeneous differential equation are: 1) the method of undetermined coefficients and 2) the method of variation of parameters.Method of Undetermined CoefficientsLet’s first break down the steps for the method of undetermined coefficients and know when it’s best to use this technique. The undetermined coefficient method works best when the right-hand side of our non-homogeneous differential equation is a function that can be written as sum or product of the following functions: x^n, e^{ax},$\sin \beta x$, or $\cos \beta x$.\begin{aligned}y^{\prime \prime} +ay^{\prime} + by &= g(x)\end{aligned}Once we identify the form of the $g(x)$, use a strategic guess for the particular solution, y_p. Let us show you some common examples for a smart guess given an expression for $g(x)$:Example of $\boldsymbol{g(x)}$Particular Solution ($\boldsymbol{y_p}$)\begin{aligned}g(x) = 2x^2\end{aligned}\begin{aligned}y_p = ax^2 + bx + c\end{aligned}\begin{aligned}g(x) = 3xe^x\end{aligned}\begin{aligned}y_p = axe^x + bex^x c\end{aligned}\begin{aligned}g(x) = x + 4\cos 3x\end{aligned}\begin{aligned}y_p = (ax + b) + c \sin 3x + d + \cos 3x\end{aligned}These three equations should give you an idea of how we come up with particular solutions we can use to complete the general solution for our second order homogeneous differential equation. Find the first and second derivatives of y_p and use them inside the equation then solve for the values of the constants.To understand how this method works, let’s try to solve the non-homogeneous differential equations, $y^{\prime \prime} + 6y^{\prime} + 5y = 4x$. First, let’s find y_h, the general solution of the equation’s associated homogeneous differential equation. \begin{aligned}y^{\prime \prime} + 6y^{\prime} + 5y = 0\end{aligned}This means that the characteristic equation (or auxiliary equation) is a quadratic equation, $r^2 + 6r + 5= 0$. The roots of this equation are $r_1 = -1$ and $r_2 = -5$, so the general form of y_h is equal to the equation shown below.\begin{aligned}y_h = C_1 e^{-x} + C_2 e^{-5x} \end{aligned}Since our equation’s right-hand side is equal to $g(x) = 4x$, the general form of our particular solution would be equivalent to $y_p = ax + b$. This means that $y_p^{\prime } = a$ and $y_p^{\prime } = 0$. We’ve learned in the past that the particular solution should be a solution to the differential equation, so substitute these expressions and solve for a and b.\begin{aligned}y^{\prime \prime}_p + 6y^{\prime}_p + 5y_p &= 4x\\0 + 6a + 5(ax +b) &= 4x\\5ax + (6a + b) &= 4x\\\\5a= 4, 6a &+b =0\\a = \dfrac{4}{5} , b =& -\dfrac{24}{5}\end{aligned}This means that $y_p = \dfrac{4}{5}x – \dfrac{24}{5}$. Now that we have both components, y_h and y_p, let’s complete the general solution of our second order non-homogeneous differential equation is:\begin{aligned}y_h&= C_1 e^{-x} + C_2 e^{-5x} \\y_p&=\dfrac{4}{5}x – \dfrac{24}{5}\\\\ \boldsymbol{y(x)} &\boldsymbol{=} \boldsymbol{C_1 e^{-x} + C_2 e^{-5x}+\dfrac{4}{5}x – \dfrac{24}{5}} \end{aligned}We’ve now shown you how the first method works, so let’s move on to our second technique: the method of variation of parameters.Method of Variation ParametersFor more complex expressions for $g(x)$, it may be difficult to come up with a particular solution that we can easily use. When that happens, it is much better for us to apply the method of variation parameters, where we assume that the general and particular solution components have the same forms. Below are the guidelines to remember when using this method:Find the expression for $y_h = C_1y_1 + C_2y_2$, using previous techniques.Write down the particular solution, $y_p = ay_1 + by_2$, based on the form of y_h.Set up the system of equations to solve for $a^{\prime}$ and $b^{\prime}$.\begin{aligned}a^{\prime}y_1 + b^{\prime}y_2 &= 0\\a^{\prime}y_1^{\prime} + b^{\prime}y_2^{\prime} &= g(x) \end{aligned}Integrate the resulting expressions for $a^{\prime}$ and $b^{\prime}$ to find a and b.Write down the general solution for the non-homogeneous differential equation: $y = y_h + y_p$.These steps are straightforward but can be complex depending on the resulting expressions. Just apply the appropriate techniques learned in the past to find the solutions using variations of parameters. Our first example below shows how we can use this method to solve more interesting non-homogeneous differential equations.Example 1Find the general solution of the non-homogeneous differential equation, $y^{\prime\prime} +y = \tan x$.SolutionSince we don’t have a guiding rule for $g(x) = \tan x$, we can’t use the method of undetermined coefficients. We can instead use the second method beginning with finding the general solution for the associated homogeneous equations. This means that the characteristic equation is equal to $r^2 + 1= 0 \rightarrow r = \pm i$, so the homogeneous solution is equal to\begin{aligned}y_h = C_1\cos x + C_2 \sin x\end{aligned}Using the form of y_h, let’s use the particular solution, $y_p = a\cos x +b\sin x$. Write down the system of linear equations:\begin{aligned}a^{\prime}\cos x+ b^{\prime}\sin x &= 0\\-a^{\prime}\sin x+ b^{\prime}\cos x &= \tan x \end{aligned}Apply the elimination method by multiplying appropriate factors then find a.\begin{aligned} {\color{blue}\sin x}(a^{\prime}\cos x+ b^{\prime}\sin x) &= {\color{blue}\sin x}(0)\\ {\color{blue}\cos x}(-a^{\prime}\sin x+ b^{\prime}\cos x) &= {\color{blue}\cos x}(\tan x)\\\\a^{\prime}\sin x\cos x + b^{\prime}\sin^2x&= 0\\-a^{\prime}\sin x\cos x + b^{\prime}\cos^2x&= \sin x \end{aligned}Simplify the equation further and you should come up with $b^{\prime} = \cos x – \sec x $. This eventually leads to $a^{\prime} = -\dfrac{\sin^2 x}{\cos x} = \cos x – \sec x$. Integrate the expressions to find the constants for y_p.\begin{aligned}\boldsymbol{a^{\prime} = \cos x – \sec x}\end{aligned}\begin{aligned}\boldsymbol{\sin x}\end{aligned}\begin{aligned} a^{\prime} &= \int (\cos x – \sec x) dx\\&= \sin x – \ln |\sec x +\tan x|\end{aligned}\begin{aligned} b^{\prime} &= \int \sin x dx\\&= -\cos x\end{aligned}\begin{aligned} y_p &= a\cos x +b\sin x\\&= \cos x(\sin x – \ln |\sec x +\tan x|) + \sin x(-\cos x)\\&= -\cos x \ln |\sec x +\tan x|\end{aligned}Combining the expressions for y_h and y_p, we have $y = y_h + y_p = C_1\cos x + C_2 \sin x -\cos x \ln |\sec x +\tan x|$.Example 2Find the general solution of the non-homogeneous differential equation, $y^{\prime\prime\prime} + 6y^{\prime\prime} + 12y^{\prime} +8y = 4x$.SolutionOur right-hand side this time is $g(x) = 4x$, so we can use the first method: undetermined coefficients. We begin by finding the general solution for the associated homogeneous equations, $y^{\prime\prime\prime} + 6y^{\prime\prime} + 12y^{\prime} +8y = 0$. This type of third-order differential equation will have a general solution of $y_h=C_1e^{r_1x}+ C_2xe^{r_2x}+ C_3x^2e^{r_3x}$, where $\{r_1, r_2, r_3\}$ are roots of the characteristic equation.

$$\begin{aligned} y_h= C_1e^{-2x}+ C_2xe^{-2x}+ C_3x^2e^{-2x}\end{aligned}$$

Since the right-hand side of the equation is equal to $4x$, the particular solution, y_p will have a general form of $ax + b$. This means that $y_p^{\prime} =a$ and $y_p^{\prime \prime} =0$.

$$\begin{aligned} y^{\prime\prime\prime} + 6y^{\prime\prime} + 12y^{\prime} + 8y &= 4x\\0 + 6(0) + 12a + 8(ax +b) &= 4x\\8ax + (12a +8b) &= 4x\\\\8a= 4 &\rightarrow a = \dfrac{1}{2}\\6+8b =0 &\rightarrow b=-\dfrac{3}{4}\end{aligned}$$

This means that we have the particular solution, $y_p = \dfrac{1}{2}x - \dfrac{3}{4}$. Combine the two components of our equation to find the general equation of the third order non-homogeneous differential equation.

$$\begin{aligned}y&= y_h + y_p\\&= C_1e^{-2x}+ C_2xe^{-2x}+ C_3x^2e^{-2x} +\dfrac{1}{2}x - \dfrac{3}{4}\end{aligned}$$

Practice Questions

1. Find the general solution of the non-homogeneous differential equation, $y^{\prime\prime} +y^{\prime}-6 = x^3$.
2. Find the general solution of the non-homogeneous differential equation, $y^{\prime\prime} -2y^{\prime} = \dfrac{e^x}{4x}$.
3. Find the general solution of the non-homogeneous differential equation, $y^{\prime\prime\prime}+3y^{\prime\prime}+3y^{\prime} +y = 2x$.